Ch 7: Checking observability of state-space models with TVPs

Sy-Miin Chow

June 4, 2021

```
#Set up display options to suppress scientific notations
rm(list=ls())
options(scipen=999)
```

Background

The goal of this script is to provide several examples of checking the observability of state-space models with time-varying parameters (TVPs). More details can be found in Chapter 7 of Gates, K., Chow, S-M., & Molenaar, P.C.M. Analysis of Intraindividual Variation. Systems Approaches to Human Process Analysis. New York, NY: Taylor & Francis.

Define a function to check observability

```
#Function to get observability matrix - see Equation 7.10 in Chapter 7
#Note that Wikipedia defines O as the transpose of Eq 7.10:
getObservabilityMatrix = function(Jacob_h,Jacob_g){
ne = dim(Jacob_g)[1]
ny = dim(Jacob_h)[1]
J = t(Jacob_h)
O_recursive = matrix(NA,ne,ny*ne)
for (i in 0:(ne-1)){
  O_recursive[,(1+i*(ny)):((i+1)*(ny))] = J
  J = t(Jacob_g)%*%J
}
return(O_recursive)
}
```

TVP Example 1

This is an example motivated by Molenaar, de Gooijer, & Schmitz, 1992. In this examples, we have 3 manifest variables, one common factor, and time-varying factor loadings:

$$y(t) = \Lambda(t)\eta(t) + \epsilon(t); y(t) = [y_1(t), y_2(t), y_3(t)]'$$

$$\Lambda(t) = [l_{11}(t), l_{21}(t), l_{31}(t)]'$$

The TV factor loadings follow the random walk model as:

$$\Lambda(t) = \Lambda(t-1) + \zeta_\Lambda(t)$$

The expanded form of this model is obtained by inserting $\Lambda(t)$ into:

$$\eta^*(t) = [\eta(t), l_{11}(t), l_{21}(t), l_{31}(t)]'.$$

Let:

$y_1(t) = l_{11}(t) * \eta(t)$ be defined as h_1,

$y_2(t) = l_{21}(t) * \eta(t)$ be defined as h_2,

$y_3(t) = l_{31}(t) * \eta(t)$ be defined as h_3.

At the dynamic level, let $g = [g_1, g_2, g_3, g_4]'$ be a vector of nonlinear dynamic functions, with:

$\eta(t) = \alpha_1 * \eta(t-1) + \zeta_\eta(t)$ be defined as g_1,

$l_{11}(t) = l_{11}(t-1) + \zeta_{l_{11}}(t)$ be defined as g_2,

$l_{21}(t) = l_{21}(t-1) + \zeta_{l_{21}}(t)$ be defined as g_3, and

$l_{31}(t) = l_{31}(t-1) + \zeta_{l_{31}}(t)$ be defined as g_4.

$Jacob_g = \frac{\partial g}{\partial \eta^*}$ is the Jacobian matrix where the (j,k)th element of $Jacob_g$ carries partial derivative of the jth dynamic function with respect to the kth latent variable $Jacob_h = \frac{\partial h}{\partial \eta^*}$ is the Jacobian matrix where the (j,k)th element of $Jacob_h$ carries partial derivative of the jth measurement function with respect to the kth latent variable.

To check the observability of this model, we substitute arbitrary numerical values into all the unknown elements, and then check the rank of the resulting observability matrix.

```
alpha_1 = .8; eta_t = 2; l_11_t = 1; l_21_t = .9; l_31_t = 1.2
ne = 4 #length of eta*(t)
Jacob_g = matrix(c(alpha_1, 0, 0, 0,
                   0,1,0,0,
                   0,0,1,0,
                   0,0,0,1),ncol=ne,byrow=TRUE)
Jacob_h = matrix(c(l_11_t,eta_t, 0, 0,
                   l_21_t,0,eta_t,0,
                   l_31_t,0,0,eta_t),ncol=ne,byrow=TRUE)

# Call the getObservabilityMatrix function to automatically construct the observability matrix;
# here is one way to check the rank of the matrix.
qr(getObservabilityMatrix(Jacob_h,Jacob_g))$rank
```

```
## [1] 4
```

The rank of the returned observability matrix is equal to ne, which suggests that the observability matrix is of full rank. This common factor model with TV factor loadings is observable!

TVP Example 2 - Expanding Example 1 to a two-factor model

In this example, we expanded Example 1 to 6 manifest variables and two latent factors with time-varying factor loadings:

$$y(t) = \Lambda(t)\eta(t) + \epsilon(t); y(t) = [y_1(t), y_2(t), y_3(t), \ldots, y_6(t)]'$$

$$\Lambda(t) = \begin{bmatrix} l_{11}(t) & 0 \\ l_{21}(t) & 0 \\ l_{31}(t) & 0 \\ 0 & l_{42}(t) \\ 0 & l_{52}(t) \\ 0 & l_{62}(t) \end{bmatrix} \tag{1}$$

The expanded form of this model is obtained by inserting $\Lambda(t)$ into:

$$\eta^*(t) = [\eta_1(t), \eta_2(t), l_{11}(t), l_{21}(t), l_{31}(t), \ldots, l_{62}(t)]',$$

and letting the TV factor loadings follow the random walk model as:

$$y_1(t) = l_{11}(t) * \eta_1(t) \text{ be defined as } h_1$$

,

$$y_2(t) = l_{21}(t) * \eta_1(t) \text{ be defined as } h_2$$

,

$$y_3(t) = l_{31}(t) * \eta_1(t) \text{ be defined as } h_3$$

,

$$y_4(t) = l_{42}(t) * \eta_2(t) \text{ be defined as } h_4$$

.

$$y_5(t) = l_{52}(t) * \eta_2(t) \text{ be defined as } h_5$$

.

$$y_6(t) = l_{62}(t) * \eta_2(t) \text{ be defined as } h_6$$

.

At the dynamic level, let $g = [g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8]'$ be a vector of nonlinear dynamic functions, with:

$$\eta_1(t) = \alpha_{11} * \eta_1(t-1) + \alpha_{12} * \eta_2(t-1) + \zeta_{\eta_1}(t), \text{ defined as } g_1$$
$$\eta_2(t) = \alpha_{22} * \eta_2(t-1) + \alpha_{21} * \eta_1(t-1) + \zeta_{\eta_2}(t), \text{ defined as } g_2$$

$$l_{11}(t) = l_{11}(t-1) + \zeta_{l_{11}}(t) \text{ defined as } g_3$$

,

$$l_{21}(t) = l_{21}(t-1) + \zeta_{l_{21}}(t) \text{ defined as } g_4$$

,

$$l_{31}(t) = l_{31}(t-1) + \zeta_{l_{31}}(t) \text{ defined as } g_5$$

,

$$\vdots$$

$$l_{62}(t) = l_{62}(t-1) + \zeta_{l_{62}}(t) \text{ be defined as } g_8$$

.

$Jacob_g = \frac{\partial g}{\partial \eta^*}$ is the Jacobian matrix where the (j,k)th element of $Jacob_g$ carries partial derivative of the jth dynamic function with respect to the kth latent variable $Jacob_h = \frac{\partial h}{\partial \eta^*}$ is the Jacobian matrix where the (j,k)th element of $Jacob_h$ carries partial derivative of the jth measurement function with respect to the kth latent variable.

To check the observability of this model, we substitute arbitrary numerical values into all the unknown elements, and then check the rank of the resulting observability matrix.

```
alpha_11 = .8; alpha_12 = .1; alpha_21 = -.3; alpha_22 = .7

eta1_t = 2; eta2_t = -3
l_11_t = 1; l_21_t = .9; l_31_t = 1.2
l_42_t = 1; l_52_t = .8; l_62_t = 0.7

ne = 8 # length of expanded eta*(t)
Jacob_g = matrix(c(alpha_11, alpha_12, rep(0,6),
                   alpha_21, alpha_22, rep(0,6),
                   0,0,1,0,0,0,0,0,
                   0,0,0,1,0,0,0,0,
                   0,0,0,0,1,0,0,0,
                   0,0,0,0,0,1,0,0,
                   0,0,0,0,0,0,1,0,
                   0,0,0,0,0,0,0,1),ncol=ne,byrow=TRUE)
Jacob_h = matrix(c(l_11_t,0,eta1_t,0,0,0,0,0,
                   l_21_t,0,0,eta1_t,0,0,0,0,
                   l_31_t,0,0,0,eta1_t,0,0,0,
                   0,l_42_t,0,0,0,eta2_t,0,0,
                   0,l_52_t,0,0,0,0,eta2_t,0,
                   0,l_62_t,0,0,0,0,0,eta2_t
                   ),ncol=ne,byrow=TRUE)

qr(getObservabilityMatrix(Jacob_h,Jacob_g))$rank
```

```
## [1] 8
```

The rank of the returned observability matrix is equal to *ne*. So this model is observable too.

TVP Example 3 - Two-factor model, TV auto- and cross-regression weights

In this example, we have a two-factor model with time-invariant factor loadings, but TV auto- and cross-regression weights at the factor level as:

$$y(t) = \Lambda\eta(t) + \epsilon(t); y(t) = [y_1(t), y_2(t), y_3(t), \dots, y_6(t)]'$$

$$\Lambda(t) = \begin{bmatrix} l_{11} & 0 \\ l_{21} & 0 \\ l_{31} & 0 \\ 0 & l_{42} \\ 0 & l_{52} \\ 0 & l_{62} \end{bmatrix} \tag{2}$$

At the dynamic level, let $g = [g_1, g_2, g_3, g_4, g_5, g_6]'$ be a vector of nonlinear dynamic functions, with:

$\eta_1(t) = \alpha_{11}(t-1)\eta_1(t-1) + \alpha_{12}(t-1)\eta_2(t-1) + \zeta_{\eta_1}(t)$, defined as g_1

$\eta_2(t) = \alpha_{22}(t-1)\eta_2(t-1) + \alpha_{21}(t-1)\eta_1(t-1) + \zeta_{\eta_2}(t)$, defined as g_2

$\alpha_{11}(t) = \alpha_{11}(t-1) + \zeta_{\alpha_{11}}(t)$, defined as g_3,

$\alpha_{12}(t) = \alpha_{12}(t-1) + \zeta_{\alpha_{12}}(t)$, defined as g_4,

$\alpha_{21}(t) = \alpha_{21}(t-1) + \zeta_{\alpha_{21}}(t)$, defined as g_5,

$\alpha_{22}(t) = \alpha_{22}(t-1) + \zeta_{\alpha_{22}}(t)$, defined as g_6, As before, $Jacob_g = \frac{\partial g}{\partial \eta^*}$ is the Jacobian matrix where the (j,k)th element of $Jacob_g$ carries partial derivative of the jth dynamic function with respect to the kth latent variable; $Jacob_h = \frac{\partial h}{\partial \eta^*}$ is the Jacobian matrix where the (j,k)th element of $Jacob_h$ carries partial derivative of the jth measurement function with respect to the kth latent variable.

To check the observability of this model, we substitute arbitrary numerical values into all the unknown elements, and then check the rank of the resulting observability matrix.

```
a_11_t = .8; a_12_t = .1; a_21_t = -.3; a_22_t = .7

eta1_t = 2; eta2_t = -3
l_11 = 1; l_21 = .9; l_31 = 1.2
l_42 = 1; l_52 = .8; l_62 = 0.7

ne = 6 # length of expanded eta*(t)
Jacob_g = matrix(c(a_11_t, a_12_t, rep(0,4),
                   a_21_t, a_22_t, rep(0,4),
                   0,0,1,0,0,0,
                   0,0,0,1,0,0,
                   0,0,0,0,1,0,
                   0,0,0,0,0,1),ncol=ne,byrow=TRUE)
Jacob_h = matrix(c(l_11,0,rep(0,4),
                   l_21,0,rep(0,4),
                   l_31,0,rep(0,4),
                   0,l_42,rep(0,4),
                   0,l_52,rep(0,4),
                   0,l_62,rep(0,4)
                   ),ncol=ne,byrow=TRUE)

qr(getObservabilityMatrix(Jacob_h,Jacob_g))$rank
```

```
## [1] 2
```

This model is not observable.

TVP Example 4 - Two-factor model, TV cross-regression weights

Seeing that the model in Example 3 is not observable, we modify it so only the two cross-regression parameters were allowed to be TVPs:

$$\eta_1(t) = \alpha_{11}\eta_1(t-1) + \alpha_{12}(t-1)\eta_2(t-1) + \zeta_{\eta_1}(t), \text{ defined as } g_1$$

$$\eta_2(t) = \alpha_{22}\eta_2(t-1) + \alpha_{21}(t-1)\eta_1(t-1) + \zeta_{\eta_2}(t), \text{ defined as } g_2$$

$$\alpha_{12}(t) = \alpha_{12}(t-1) + \zeta_{\alpha_{12}}(t), \text{ defined as } g_3$$

,

$$\alpha_{21}(t) = \alpha_{21}(t-1) + \zeta_{\alpha_{21}}(t), \text{ defined as } g_4$$

.

Now let's repeat the same process.

```r
a_11 = .8; a_12_t = .1; a_21_t = -.3; a_22 = .7

eta1_t = 2; eta2_t = -3
l_11 = 1; l_21 = .9; l_31 = 1.2
l_42 = 1; l_52 = .8; l_62 = 0.7

ne = 4 # length of expanded eta*(t)
Jacob_g = matrix(c(a_11, a_12_t, eta1_t, 0,
                   a_21_t, a_22, 0, eta2_t,
                   0,0,1,0,
                   0,0,0,1),ncol=ne,byrow=TRUE)
Jacob_h = matrix(c(l_11,0,rep(0,2),
                   l_21,0,rep(0,2),
                   l_31,0,rep(0,2),
                   0,l_42,rep(0,2),
                   0,l_52,rep(0,2),
                   0,l_62,rep(0,2)
),ncol=ne,byrow=TRUE)

qr(getObservabilityMatrix(Jacob_h,Jacob_g))$rank
```

```
## [1] 4
```

This streamlined model is now observable.